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LETTER TO THE EDITOR

Exact travelling wave solutions of nonlinear evolution equation of surface waves in a convecting fluid

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Abstract. Exact travelling wave solutions in terms of the Weierstrass $\wp$-function of the equation $\eta_t+a_o\eta_x+a_1\eta\eta_x+a_2\eta_{xxx}+b_o\eta_{xx}+b_1(\eta\eta_x)_x+b_2\eta_{xxxx}=0$ are obtained. These solutions may describe particularly both the evolution of bounded conoidal wave and kink-shaped waves.

In this letter exact travelling wave solutions of nonlinear evolution equation of surface waves in a convecting fluid

$$\eta_t+a_0\eta_x+a_1\eta\eta_x+a_2\eta_{xxx}+b_0\eta_{xx}+b_1(\eta\eta_x)_x+b_2\eta_{xxxx}=0 \tag{1}$$

will be obtained. This equation has been derived in [1, 2] to describe oscillatory Rayleigh–Marangoni instability in a liquid layer with free boundaries. When $b_1=b_2=0$ equation (1) corresponds to the well known Burgers–Korteveg–de Vries (BKdV) equation used, in particular, in the theory of nonlinear waves on viscous fluids [3]. Case $a_1=a_2=0$ leads to the equation that has been proposed in [4] to describe free surface waves caused by the Marangoni instability in a liquid layer with rigid lower boundary. The Kuramoto–Sivashinsky (KS) equation [5–7] arises when $a_2=b_1=0$. Finally, the general wave dynamics equation, derived in [6] to describe nonlinear surface waves on viscous fluid moving down on inclined plane, corresponds to the situation $b_1=0$.

Exact solutions of the BKdV equation were obtained independently in [8–10]. A travelling wavefront solution of the KS equation was derived in [5, 8, 11]. In [4] a proposal for a stationary solution was expressed by the Jacobi elliptic function *cn* for the case $a_1=a_2=0$. The situation $b_1=0$ was studied in detail in [8], and some exact travelling wave solutions were obtained. Finally, two travelling solitary wave solutions of equation (1) when $a_0=0$ were found in [12].

Following the Weiss–Tabor–Carnevalle method [13], the solutions of equation (1) for non-vanishing a_i may be sought expanding the dependent variable in a Laurent series about the pole manifold $F(x, t)=0$:

$$\eta=F^{-2}\sum_{j=0}^{\infty}\eta_jF^j. \tag{2}$$

One can show that equation (1) does not possess the Painlevé property for arbitrary F. However, series (2) may be truncated at the third term, which leads to the following auto-Bäcklund transformation for the function η:

$$\eta = \frac{12b_2}{b_1}(\ln F)_{xx} + \frac{12}{5b_1}\left(a_2 - \frac{a_1 b_2}{b_1}\right)(\ln F)_x + \tilde{\eta} \tag{3}$$

if function $\tilde{\eta}$ is a solution of equation (1) and function F satisfies some overdetermined system of equations. Solving this system one can obtain some exact travelling wave solutions of (1). A similar procedure was applied in [8] to derive solitary wave solutions of the KS equation as well as the general wave dynamic wave equation. However, another method will be used here in order to obtain more general form solutions. Recently a method was developed [9], based on seeking exact solutions of a wide class of second-order nonlinear ordinary differential equations (ODE) in terms of the Weierstrass function $\wp$, and it was assumed that ODE of higher order may be solved similarly. Concerning only travelling wave solutions of the equation (1), one can reduce it to the third-order ODE of the form:

$$b_2\eta_{\theta\theta\theta} + a_2\eta_{\theta\theta} + b_0\eta_\theta + b_1\eta\eta_\theta + \frac{a_1}{2}\eta^2 + (a_0 - V)\eta + P = 0 \tag{4}$$

where $\theta = x - Vt$, P is a constant. The possible solution of the (1) may contain simple and second-order poles, as it results from the auto-Bäcklund transformation (3). Therefore one can propose the following functional form of solution in terms of function $\wp(\theta, g_2, g_3)$:

$$\eta = A\wp + B\frac{\wp_\theta}{\wp + C} + D \tag{5}$$

where A, B, C, D are parameters. Substituting (5) into (4) and equating coefficients at each order of $\wp$ and $\wp_\theta$ to zero one can derive a system of algebraic equations in A, B, C, D, phase velocity V and Weierstrass function parameters g_2, g_3:

$$B(g_2C - g_3 - 4C^3) = 0 \qquad B(12C^2 - g_2) = 0$$

$$P = (V - a_0)D + 4a_1B^2C - \frac{a_1}{2}D^2 - \frac{a_2}{2}g_2A - 2b_0BC + 12b_2BC^2$$

$$-b_1\left(2ABC^2 + \frac{g_2}{2}AB - 2BCD\right)$$

$$(a_0 - V)A + a_1(2B^2 + AD) + 2b_0B + 2b_1(BD - ABC) = 0$$

$$a_1A^2 + 12a_2A + 24b_2B + 12b_1AB = 0$$

$$B(a_0 - V + a_1(D - AC)) = 0 \qquad A(12b_2 + b_1A) = 0$$

$$a_1AB + 2a_2B + b_0A + b_1(AD + 2b_1B^2) = 0.$$

The solutions of these equations are:

(i) $\quad A = -\frac{12b_2}{b_1}, B = 0 \qquad D = -\frac{b_0}{b_1} \qquad V = a_0 - \frac{a_1 b_0}{b_1}$

g_2 and g_3 are free parameters, if $a_2 = a_1 b_2/b_1$;

(ii) $$A = -\frac{12b_2}{b_1} \qquad B = -\frac{6}{5b_1}\left(a_2 - \frac{a_1 b_2}{b_1}\right)$$

$$D = -\frac{b_0}{b_1} + \frac{a_1}{b_1^2}\left(a_2 - \frac{a_1 b_2}{b_1}\right) + \frac{1}{25 b_1 b_2}\left(a_2 - \frac{a_1 b_2}{b_1}\right)^2$$

$$V = a_0 - \frac{a_1 b_0}{b_1} + \frac{12 a_1 b_2}{b_1} C + \frac{a_1^2}{b_1^2}\left(a_2 - \frac{a_1 b_2}{b_1}\right)$$

$$+ \frac{a_1}{25 b_1 b_2}\left(a_2 - \frac{a_1 b_2}{b_1}\right)^2 \qquad g_2 = 12C^2 \qquad g_3 = 8C^3$$

when either

$$C = -\frac{1}{300 b_2^2}\left(a_2 - \frac{a_1 b_2}{b_1}\right)^2 \tag{6}$$

or C is a free parameter, if $a_2 = 6a_1 b_2/b_1$.

The solution (5) with parameters defined by (i) may describe a particular bounded cnoidal wave, propagating with fixed phase velocity, of the form:

$$\eta = \frac{12b_2}{b_1} k^2 \kappa^2 cn^2(k\theta, \kappa) - \frac{b_0}{b_1} - \frac{4b_2(2\kappa^2 - 1)}{b_1} \tag{7}$$

where k is a free parameter, κ is the Jacobi elliptic functions modulus. When $\kappa = 1$, solution (7) corresponds to the travelling solitary wave solution found recently in [12].

The solution (5) with parameters defined by (ii) may describe a bounded travelling kink-shaped wave:

$$\eta = -\frac{36 b_2 C}{b_1} \cosh^{-2}(\sqrt{-3C}\theta) + \frac{12}{5b_1}\left(a_2 - \frac{a_1 b_2}{b_1}\right)\sqrt{-3C} \tanh \sqrt{-3C}\theta) +$$

$$-\frac{b_0}{b_1} + \frac{a_1}{b_1^2}\left(a_2 - \frac{a_1 b_2}{b_1}\right) + \frac{1}{25 b_1 b_2}\left(a_2 - \frac{a_1 b_2}{b_1}\right)^2 + \frac{12 b_2 C}{b_1}. \tag{8}$$

When parameter C is defined by (6) we have two known kink-shaped solutions for positive and negative values of $\sqrt{-3C}$ [12]. When C is a free parameter, a new kink-shaped solution (8) arises. Its main feature is that this wave may propagate with any phase velocity value.

Besides bounded solutions (7) and (8), solution (5) may describe also unbounded ones in the form of localized and periodic discontinuities. The forms of these solutions may be simply presented using well known relationships between the Weierstrass $\wp$-function and Jacobi elliptic functions.

Finally it is to be noted that the functional form (5) of the solution of equation (1) in terms of the Weierstrass $\wp$-function is not unique. Using the above mentioned procedure one can show that there exist at least one other solution of the form:

$$\eta = -\frac{12}{25 b_1 b_2}\left(a_2 - \frac{a_1 b_2}{b_1}\right)^2 \exp(2y)\wp(\exp(y) + G, 0, g_3) \tag{9}$$

where

$$y = \exp(\gamma\theta) \qquad \gamma = -\frac{1}{5b_2}\left(a_2 - \frac{a_1 b_2}{b_1}\right)$$

G and g_3 are free parameters. This solution allows us to describe only one bounded kink-shaped wave solution of a form coinciding with (8) for parameter C defined by (6) and for negative value of $\sqrt{-3C}$. However, periodic discontinuities described by the solution (9) differ from the ones described by the solution (5).

To sum up, two exact travelling wave solutions (5) and (9) of equation (1) are found, that allow us to obtain a new cnoidal wave solution (kink-shaped solution) propagating with any phase velocity, as well as unbounded solutions in the form of localized and periodical discontinuities.

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